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On the Representation of Extremal Functions in the L^1-Norm

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1. INTRODUCTION

Bernstein (see [1, p. 249–254]) studied the following problem:

(a) Among all rational functions of the form

$$\frac{x^n + \sum_{i=0}^{n-1} a_i x^i}{p_l(x)}, \qquad x \in [-1, +1], \quad (a_0, ..., a_{n-1}) \in \mathbb{R}^n,$$

where p_l is a real fixed polynomial of degree l, $n \geqslant l$, which is positive in the interval $[-1, +1]$, to find that one which has the least deviation from zero in the L^1-norm.

In this paper we consider the following problem, which can be regarded as a generalization of (a):

(b) Let T_n/s_l be that function, which has the least deviation from zero in the L^1-norm among all functions of the form

$$\frac{\sum_{i=n-k}^{n} (A_i \cos i\varphi + B_i \sin i\varphi) + \sum_{i=0}^{n-k-1} (a_i \cos i\varphi + b_i \sin i\varphi)}{s_l(\varphi)},$$

where $\varphi \in [-\pi, \pi]$ and $A_n, ..., A_{n-k}, B_n, ..., B_{n-k} \in \mathbb{R}$ are given, $(a_0, ..., a_{n-k-1}, b_0, ..., b_{n-k-1}) \in \mathbb{R}^{2n-2k}$, and s_l is a fixed trigonometric polynomial of degree l, $n \geqslant l + k + 1$, with real coefficients, which is positive in the interval $[-\pi, +\pi)$. How can the extremal function T_n/s_l be represented?

Problem (b) for $k = 0$ in the Chebyshev norm was solved by Szegö [10].

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2. On the Problem of Szegö in the L^1-Norm

Definition 1 (see [10]). Let s_l be a trigonometric polynomial of degree l with real coefficients which is positive on $[-\pi, +\pi)$. Then s_l can be represented in the form

$$s_l(\varphi) = \gamma^2 \mid g_l(z)\mid^2,$$

where $\gamma \in \mathbb{R}^+$, $g_l(z) = \prod_{\nu=1}^{l} (z - z_\nu)$, $z_\nu \in \{z \in \mathbb{C} \mid \mid z \mid < 1\}$, $z = e^{i\varphi}$, $\varphi \in [-\pi, +\pi]$. We define for $A, B \in \mathbb{R}$, $A^2 + B^2 > 0$, $n \in \mathbb{N}_0$,

$$\gamma^2 T_n(\varphi, s_l) := \operatorname{Re} \left\{(A - iB)\, z^{n-2l} \frac{g_l(z)}{\overline{g_l(z)}}\right\} s_l(\varphi)$$

for $z = e^{i\varphi}$, $\varphi \in [-\pi, +\pi]$.

If $n \geqslant l + 1$, then $T_n(\varphi, s_l)$ is a trigonometric polynomial of degree n with real coefficients and is of the form $A \cos n\varphi + B \sin n\varphi + \cdots$.

Notation. In the following let $\Psi_n(z) = z^{n-2l}(g_l(z)/\overline{g_l(z)})$ and $g_l^*(z) = z^l \bar{g}_l(z^{-1}) = \prod_{\nu=1}^{l} (1 - \bar{z}_\nu z)$, the reciprocal polynomial of $g_l(z)$.

Definition 2. Let $a = x_0 < x_1 < \cdots < x_r = b$, $r \in \mathbb{N}$, be a decomposition of the interval $[a, b]$. We say that a function v defined on $[a, b]$ is a sign function on $[a, b]$, if either v or $-v$ takes the value $(-1)^j$ on the interval (x_{j-1}, x_j), $j = 1,..., r$.

It is easy to see that the following lemma is valid.

Lemma 1. *If v is a sign function on $[-\pi, +\pi]$, then, for $k \in \mathbb{N}_0$,*

$$\int_{-\pi}^{+\pi} e^{-ik\varphi} v(\varphi)\, d\varphi = \overline{\int_{-\pi}^{+\pi} e^{ik\varphi} v(\varphi)\, d\varphi}.$$

The following theorem now gives us the solution of the problem of Szegö in the L^1-norm. Concerning the methods used in the proof of Theorem 1, we refer to [1, p. 252] and [3].

Theorem 1. *Let $n \geqslant l + 1$.*

(a) $$\int_{-\pi}^{+\pi} \frac{\begin{Bmatrix}\sin k\varphi \\ \cos k\varphi\end{Bmatrix}}{s_l(\varphi)} \operatorname{sgn} T_n(\varphi, s_l)\, d\varphi = 0, \qquad k \in \{0,..., n - 1\}.$$

(b) *If $S_n(\varphi) = A \cos n\varphi + B \sin n\varphi + \sum_{i=0}^{n-1} (a_i \cos i\varphi + b_i \sin i\varphi)$, $S_n \neq T_n(\cdot, s_l)$ is a trigonometric polynomial, then*

$$\int_{-\pi}^{+\pi} \frac{\mid S_n(\varphi)\mid}{s_l(\varphi)}\, d\varphi > \int_{-\pi}^{+\pi} \frac{\mid T_n(\varphi, s_l)\mid}{s_l(\varphi)}\, d\varphi = \frac{4}{\gamma^2} (A^2 + B^2)^{1/2}.$$

Proof. (a) Since $1 = \overline{\Psi_n(z)}\,\Psi_n(z) = |\Psi_n(z)|^2$ for $z = e^{i\varphi}$, $\varphi \in [-\pi, +\pi]$, there exists a real function ϕ such that $e^{i\phi(\varphi)} = \Psi_n(z)$ $(z = e^{i\varphi}, \varphi \in [-\pi, +\pi])$. Therefore

$$\gamma^2 \frac{T_n(\varphi, s_l)}{s_l(\varphi)} = \mathrm{Re}\{(A - iB)\, e^{i\phi(\varphi)}\} = (A^2 + B^2)^{1/2} \cos(\phi(\varphi) + \alpha), \quad (1)$$

where $e^{i\alpha} = (A + iB)/(A^2 + B^2)^{1/2}$.

Considering the expansion

$$\mathrm{sgn}\, \cos(\phi + \alpha) = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r + 1} \cos(2r + 1)(\phi + \alpha),$$

where $(z = e^{i\varphi}, \varphi \in [-\pi, +\pi])$

$$\cos(2r + 1)(\phi(\varphi) + \alpha) = \mathrm{Re}\{[e^{i\alpha}\Psi_n(z)]^{2r+1}\}, \quad (2)$$

we have

$$\int_{-\pi}^{+\pi} \frac{e^{-ik\varphi}}{s_l(\varphi)}\, \mathrm{sgn}\, \cos(\phi(\varphi) + \alpha)\, d\varphi = \frac{4}{\pi\gamma^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r + 1} I_{r,k},$$

with $(z = e^{i\varphi})$ $I_{r,k} = \int_{-\pi}^{+\pi} (z^{-k}/g_l(z)\,\overline{g_l(z)})\, \mathrm{Re}\{[e^{i\alpha}\Psi_n(z)]^{2r+1}\}\, d\varphi$.

An elementary calculation gives

$$\begin{aligned} I_{r,k} &= \frac{1}{2} \int_{-\pi}^{+\pi} \frac{z^{-k}}{g_l(z)\,\overline{g_l(z)}} [e^{i\alpha}\Psi_n(z)]^{2r+1}\, d\varphi \\ &\quad + \frac{1}{2} \int_{-\pi}^{+\pi} \frac{z^{k}}{g_l(\bar z)\,\overline{g_l(\bar z)}} [e^{-i\alpha}\overline{\Psi_n(\bar z)}]^{2r+1}\, d\varphi \\ &= \frac{e^{i(2r+1)\alpha}}{2} \int_{|z|=1} z^{-k+l+(n-l)(2r+1)} \frac{[g_l(z)]^{2r}}{[g_l^*(z)]^{2r+2}} \frac{dz}{iz} \\ &\quad + \frac{e^{-i(2r+1)\alpha}}{2} \int_{|z|=1} z^{k+l+(n-l)(2r+1)} \frac{[\bar g_l(z)]^{2r}}{[\bar g_l^*(z)]^{2r+2}} \frac{dz}{iz}. \end{aligned}$$

Since $|z_\nu| < 1$ for $\nu = 1,\ldots, l$ and $-k - 1 + l + (n - l)(2r + 1) \geqslant 0$ for $k \in \{0,\ldots, n - 1\}$, $r \in \mathbb{N}_0$, both integrands are analytic in the unit disk.

By Cauchy's theorem,

$$I_{r,k} = 0 \qquad \text{for} \quad k \in \{0,\ldots, n - 1\}, \quad r \in \mathbb{N}_0.$$

Part (a) of Theorem 1 follows now from Lemma 1.

Concerning (b), it follows from (a) and Lemma 4.-4 of [8, p. 103] that

$T_n(\varphi, s_l)/s_l(\varphi)$ is the unique extremal function. From the proof of (a) we obtain with the aid of (1)

$$\int_{-\pi}^{+\pi} \left| \frac{T_n(\varphi, s_l)}{s_l(\varphi)} \right| d\varphi = \frac{(A^2 + B^2)^{1/2}}{\gamma^2} \int_{-\pi}^{+\pi} |\cos(\phi(\varphi) + \alpha)|\, d\varphi. \tag{3}$$

Using the formula

$$|\cos(\phi(\varphi) + \alpha)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{(-1)^r}{4r^2 - 1} \cos 2r(\phi(\varphi) + \alpha)$$

and, as is easy to see by (2),

$$\int_{-\pi}^{+\pi} \cos 2r(\phi(\varphi) + \alpha)\, d\varphi = 0 \qquad \text{for} \quad r \in \mathbb{N},$$

we have

$$\int_{-\pi}^{+\pi} |\cos(\phi(\varphi) + \alpha)|\, d\varphi = \frac{2}{\pi} \int_{-\pi}^{+\pi} d\varphi = 4.$$

The result now follows from (3).

3. Representation of the Extremal Function of Problem (b)

Definition 3. Assume $n, l, m \in \mathbb{N}_0$, and let

$$\prod_{\nu=1}^{m} (z - d_\nu)^2 = \sum_{\mu=0}^{2m} a_\mu z^\mu + i \sum_{\mu=0}^{2m-1} b_\mu z^\mu,$$

where $d_\nu \in \{z \in \mathbb{C} \mid |z| < 1\}$, $a_\mu, b_\mu \in \mathbb{R}$. Let $\prod_{\nu=1}^{m} (z - d_\nu)^2 = 1$ for $m = 0$. We define for $n \geqslant l + m + 1$

$$\mathscr{T}_{n,s_l}\left(\varphi, \cdot \prod_{\nu=1}^{m} (z - d_\nu)\right) := \sum_{\mu=0}^{2m} a_\mu T_{n-2m+\mu}(\varphi, s_l) + \sum_{\mu=0}^{2m-1} b_\mu T'_{n-2m+\mu}(\varphi, s_l),$$

where

$$\gamma^2 T'_n(\varphi, s_l) := \operatorname{Re} \left\{(B + iA)\, z^{n-2l} \frac{g_l(z)}{\overline{g_l(z)}}\right\} s_l(\varphi)$$

for $z = e^{i\varphi}$, $\varphi \in [-\pi, +\pi]$.

Notation. For $a, b \in \mathbb{R}$, f: $[a, b] \to \mathbb{R}$ we denote the number of sign changes of f on (a, b) by $S^-(f)$.

THEOREM 2. (a) $\mathscr{T}_{n,s_l}(\cdot, \prod_{\nu=1}^{m}(z-d_\nu))$ *is a trigonometric polynomial of degree n with real coefficients of the form* $A\cos n\varphi + B\sin n\varphi + \cdots$,

$$\text{(b)}\quad \int_{-\pi}^{+\pi} \frac{\begin{Bmatrix}\sin k\varphi\\ \cos k\varphi\end{Bmatrix}}{s_l(\varphi)} \operatorname{sgn} \mathscr{T}_{n,s_l}\left(\varphi, \prod_{\nu=1}^{m}(z-d_\nu)\right) d\varphi = 0,$$

$$k \in \{0,\dots, n-m-1\},$$

$$\text{(c)}\quad S^-\left(\mathscr{T}_{n,s_l}\left(\cdot, \prod_{\nu=1}^{m}(z-d_\nu)\right)\right) \geqslant 2n-1,$$

(d) *If* $3m \leqslant 2n-2l-1$, *then with* $\prod_{\nu=1}^{m}(z-d_\nu) = \sum_{k=0}^{m} c_k z^k$ *one has*

$$\int_{-\pi}^{+\pi} \left| \frac{\mathscr{T}_{n,s_l}(\varphi, \prod_{\nu=1}^{m}(z-d_\nu))}{s_l(\varphi)} \right| d\varphi = \frac{4}{\gamma^2}(A^2+B^2)^{1/2} \sum_{k=0}^{m} |c_k|^2.$$

Proof. Let $h_m(z) := \prod_{\nu=1}^{m}(z-d_\nu)$. Concerning part (a), one has, for $z = e^{i\varphi}$, $\varphi \in [-\pi, +\pi]$,

$$\begin{aligned}\gamma^2 \mathscr{T}_{n,s_l}&\left(\varphi, \prod_{\nu=1}^{m}(z-d_\nu)\right)\\ &= \operatorname{Re}\{(A-iB)\,\Psi_{n-2m} h_m^2(z)\}\, s_l(\varphi)\\ &= \operatorname{Re}\left\{(A-iB)\, z^{n-2l-2m} \frac{g_l(z)\, h_m(z)}{\overline{g_l(z)}\ \overline{h_m(z)}}\right\} s_l(\varphi)\, |h_m(z)|^2. \end{aligned} \tag{1}$$

Hence

$$\gamma^2 \mathscr{T}_{n,s_l}\left(\varphi, \prod_{\nu=1}^{m}(z-d_\nu)\right) = T_n(\varphi, s_l\, |h_m|^2). \tag{2}$$

(b) From (2) and Theorem 1(a), it follows that

$$\int_{-\pi}^{+\pi} \frac{\begin{Bmatrix}\sin k\varphi\\ \cos k\varphi\end{Bmatrix}}{s_l(\varphi)\, |h_m(e^{i\varphi})|^2} \operatorname{sgn} \mathscr{T}_{n,s_l}\left(\varphi, \prod_{\nu=1}^{m}(z-d_\nu)\right) d\varphi = 0 \tag{3}$$

for $k \in \{0,\dots, n-1\}$. Noting that $|h_m(e^{i\varphi})|^2$ is a trigonometric polynomial of degree m, Theorem 1(b) is proved.

(c) The assertion follows from (3) and Lemma 4.-6 of [8, p. 108].

(d) As in the proof of Theorem 1, one demonstrates the existence of a real function ϕ such that

$$(A^2+B^2)^{1/2}\cos(\phi(\varphi)+\alpha) = \operatorname{Re}\left((A-iB)\, z^{n-2l-2m} \frac{g_l(z)\, h_m(z)}{\overline{g_l(z)}\ \overline{h_m(z)}}\right),$$

where $e^{i\alpha} = (A + iB)/(A^2 + B^2)^{1/2}$. Therefore with (1),

$$\int_{-\pi}^{+\pi} \left| \frac{\mathcal{T}_{n,s_l}(\varphi, \prod_{\nu=1}^{m} (z - d_\nu))}{s_l(\varphi)} \right| d\varphi$$
$$= \frac{(A^2 + B^2)^{1/2}}{\gamma^2} \int_{-\pi}^{+\pi} |\cos(\phi(\varphi) + \alpha)|\, |h_m(e^{i\varphi})|^2\, d\varphi. \tag{4}$$

Since $3m \leqslant 2n - 2l - 1$, it follows, by Cauchy's theorem, that

$$\int_{-\pi}^{+\pi} e^{-ik\varphi} \cos 2r(\phi(\varphi) + \alpha)\, d\varphi$$
$$= \int_{|z|=1} z^{-k+(n-l-m)2r} \left[\frac{g_l(z)\, h_m(z)}{g_l^*(z)\, h_m^*(z)}\right]^{2r} \frac{dz}{iz}$$
$$+ \int_{|z|=1} z^{k+(n-l-m)2r} \left[\frac{\bar{g}_l(z)\, \bar{h}_m(z)}{\bar{g}_l^*(z)\, \bar{h}_m^*(z)}\right]^{2r} \frac{dz}{iz} = 0$$

for $k \in \{0,\ldots, m\}$, $r \in \mathbb{N}$. Analogously one shows that

$$\int_{-\pi}^{+\pi} e^{ik\varphi} \cos 2r(\phi(\varphi) + \alpha)\, d\varphi = 0 \qquad \text{for} \quad k \in \{0,\ldots, m\}, \quad r \in \mathbb{N}. \tag{5}$$

With the help of the Fourier expansion of $|\cos(\phi + \alpha)|$ and (5), we obtain

$$\int_{-\pi}^{+\pi} |\cos(\phi(\varphi) + \alpha)|\, |h_m(e^{i\varphi})|^2\, d\varphi = \frac{2}{\pi} \int_{-\pi}^{+\pi} |h_m(e^{i\varphi})|^2\, d\varphi.$$

In view of Parseval's formula and (4), Theorem 2(d) is proved.

Now the question arises whether every trigonometric polynomial of degree n with property (b) of Theorem 2 is a polynomial of the form $\mathcal{T}_{n,s_l}(\varphi, \prod_{\nu=1}^{m} (z - d_\nu))$. It will be shown that this is valid under appropriate conditions.

The following lemma is known (see, e.g., [7]); λ denotes the Lebesgue measure.

LEMMA 2. (a) *Let v be a bounded function on $[-\pi, +\pi]$ with at most a finite number of discontinuities. If $S^-(v) \leqslant 2n - 2$ and*

$$\int_{[-\pi,+\pi]} \frac{\begin{Bmatrix}\sin k\varphi\\ \cos k\varphi\end{Bmatrix}}{s_l(\varphi)}\, v(\varphi)\, d\lambda(\varphi) = 0, \qquad k \in \{0,\ldots, n - 1\},$$

then $v = 0$ λ a.e. on $[-\pi, +\pi]$.

(b) *If v, w are sign functions on $[-\pi, +\pi]$ with $S^-(v) = k$ and $S^-(w) = l$, $k, l \in \mathbb{N}_0$, then $S^-(v \pm w) \leqslant \min\{l, k\}$.*

THEOREM 3. *Let* $n \geqslant l + m + 1$, $3m \leqslant 2n - 2l - 1$. *If* S_n *is a trigonometric polynomial of degree n with leading coefficients* $A, B \in \mathbb{R}$, $A^2 + B^2 > 0$, $S^-(S_n) \geqslant 2n - 1$, *and*

$$\int_{-\pi}^{+\pi} \frac{\begin{Bmatrix}\sin k\varphi\\ \cos k\varphi\end{Bmatrix}}{s_l(\varphi)} \operatorname{sgn} S_n(\varphi)\, d\varphi = 0, \qquad k \in \{0,\dots, n-m-1\},$$

then there exists a polynomial $\prod_{\nu=1}^{m} (z - d_\nu)$, $d_\nu \in \{z \in \mathbb{C} \mid |z| < 1\}$, *such that*

$$S_n(\varphi) = \mathscr{T}_{n,s_l}\left(\varphi, \prod_{\nu=1}^{m} (z - d_\nu)\right) \qquad (\varphi \in [-\pi, +\pi]).$$

If in addition $\int_{-\pi}^{+\pi} \cos(n-m)\varphi \operatorname{sgn} S_n(\varphi)\, d\varphi \neq 0$ *or* $\int_{-\pi}^{+\pi} \sin(n-m)\varphi \operatorname{sgn} S_n(\varphi)\, d\varphi \neq 0$, *then* $d_\nu \in \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ *for* $\nu \in \{1,\dots, m\}$.

Proof. Put

$$\frac{1}{4} \int_{-\pi}^{+\pi} \frac{e^{-ik\varphi}}{g_l(e^{i\varphi})\, \overline{g_l(e^{i\varphi})}} \operatorname{sgn} S_n(\varphi)\, d\varphi = a_k, \tag{1}$$

with $a_k \in \mathbb{C}$ for $k \in \{n-m,\dots, n\}$. Furthermore, let

$$[g_l^*(z)]^2 \sum_{\mu=0}^{m} a_{n-m+\mu} z^\mu = \sum_{\mu=0}^{m} b_\mu z^\mu + \cdots. \tag{2}$$

In view of [1, p. 274–275], for the $m+1$ given numbers $b_0,\dots, b_m \in \mathbb{C}$ there exists an analytic rational function in the unit disk such that

$$L \frac{c_0 + c_1 z + \cdots + c_{m_1} z^{m_1}}{\bar{c}_0 z^{m_1} + \bar{c}_1 z^{m_1 - 1} + \cdots + \bar{c}_{m_1}} = \sum_{\mu=0}^{m} b_\mu z^\mu + \sum_{\mu=m+1}^{\infty} \gamma_\mu z^\mu$$

for $z \in \{z \in \mathbb{C} \mid |z| \leqslant 1\}$, where $L \in \mathbb{R}^+$, $m_1 \in \mathbb{N}$ and $m_1 \leqslant m$, $c_{m_1} \neq 0$, $\gamma_\mu \in \mathbb{C}$ for $\mu \in \{m+1, m+2,\dots\}$. From the equation

$$L \frac{c_0 + \cdots + c_{m_1} z^{m_1}}{\bar{c}_0 z^{m_1} + \cdots + \bar{c}_{m_1}} = L e^{i\gamma} \frac{\prod_{\nu=1}^{m_1} (z - d_\nu)}{\prod_{\nu=1}^{m_1} (1 - \bar{d}_\nu z)}$$

with $d_\nu \in \{z \in \mathbb{C} \mid |z| < 1\}$ for $\nu = 1,\dots, m_1$, $\gamma \in \mathbb{R}$, it follows from (2) that

$$L e^{i\gamma} \Omega(z) = \sum_{\mu=0}^{m} a_{n-m+\mu} z^\mu + \sum_{\mu=m+1}^{m} \gamma'_\mu z^\mu \tag{3}$$

for $z \in \{z \in \mathbb{C} \mid |z| \leqslant 1\}$, $\gamma'_\mu \in \mathbb{C}$, $\mu \in \{m+1,\dots\}$, where

$$h_{m_1}(z) := \prod_{\nu=1}^{m_1} (z - d_\nu) \qquad \text{and} \qquad \Omega(z) := \frac{h_{m_1}(z)}{[g_l^*(z)]^2\, h_{m_1}^*(z)}.$$

Therefore

$$\frac{Le^{i\gamma}}{2\pi i}\int_{|z|=1} z^{-(k+1)+(n-m)}\Omega(z)\,dz = 0 \qquad \text{for} \quad 0 \leqslant k \leqslant n-m-1,$$
$$= a_k \qquad \text{for} \quad n-m \leqslant k \leqslant n. \tag{4}$$

We consider now the trigonometric polynomial of degree $n - m + m_1$

$$G(\varphi) := \operatorname{Re}\left(e^{i\gamma} z^{n-2l-m-m_1} \frac{g_l(z)\, h_{m_1}(z)}{\overline{g_l(z)}\ \overline{h_{m_1}(z)}}\right) |\, g_l(z)\, h_{m_1}(z)|^2, \tag{5}$$

for $z = e^{i\varphi}$, $\varphi \in [-\pi, +\pi]$.

As in the proof of Theorem 1(a) one shows that

$$\int_{-\pi}^{+\pi} \frac{z^{-k}}{g_l(z)\,\overline{g_l(z)}} \operatorname{sgn} G(\varphi)\, d\varphi = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} I_{r,k} \qquad (z = e^{i\varphi}),$$

where

$$I_{r,k} = \int_{-\pi}^{+\pi} \frac{z^{-k}}{g_l(z)\,\overline{g_l(z)}} \operatorname{Re}\left\{\left[e^{i\gamma} z^{n-2l-m-m_1} \frac{g_l(z)\, h_{m_1}(z)}{\overline{g_l(z)}\ \overline{h_{m_1}(z)}}\right]^{2r+1}\right\} d\varphi.$$

We obtain for $r = 0$

$$I_{0,k} = e^{i\gamma} \int_{|z|=1} z^{n-m-k}\Omega(z)\, \frac{dz}{iz} + e^{-i\gamma} \int_{|z|=1} z^{n+k-m}\overline{\Omega}(z)\, \frac{dz}{iz}$$
$$= \frac{e^{i\gamma}}{i} \int_{|z|=1} z^{-(k+1)+(n-m)}\Omega(z)\, dz.$$

By Cauchy's theorem and the fact that $3m \leqslant 2n - 2l - 1$,

$$I_{r,k} = 0 \qquad \text{for} \quad k \in \{0,\dots, n\}, \quad r \in \mathbb{N}.$$

Hence by (4)

$$\frac{1}{4e^{i\gamma}} \int_{-\pi}^{+\pi} \frac{z^{-k}}{g_l(z)\,\overline{g_l(z)}} \operatorname{sgn} G(\varphi)\, d\varphi = \frac{1}{2\pi i} \int_{|z|=1} z^{-(k+1)+(n-m)}\Omega(z)\, dz$$
$$= 0 \qquad \text{for} \quad 0 \leqslant k \leqslant n-m-1,$$
$$= \frac{a_k}{Le^{i\gamma}} \qquad \text{for} \quad n-m \leqslant k \leqslant n.$$

From (1) and Lemma 1 it follows now that

$$\int_{-\pi}^{+\pi} \frac{\begin{Bmatrix}\sin k\varphi\\ \cos k\varphi\end{Bmatrix}}{s_l(\varphi)} (L \operatorname{sgn} G(\varphi) - \operatorname{sgn} S_n(\varphi))\, d\varphi = 0, \qquad k \in \{0,..., n\}.$$

Since

$$\begin{aligned} \operatorname{sgn}(L \operatorname{sgn} G(\varphi) - \operatorname{sgn} S_n(\varphi)) &= \operatorname{sgn} G(\varphi) && \text{for} \quad L > 1, \\ &= -\operatorname{sgn} S_n(\varphi) && \text{for} \quad L < 1, \end{aligned}$$

Lemma 2 implies $L = 1$. Furthermore, Lemma 2(b) and 2(a) give

$$\operatorname{sgn} G(\varphi) = \operatorname{sgn} S_n(\varphi) \qquad (\varphi \in [-\pi, +\pi]). \tag{6}$$

Since S_n has at least $2n - 1$ zeros on $(-\pi, +\pi)$, G must be a polynomial of degree n; hence $m_1 = m$.

If $S^-(S_n) = 2n$, it follows immediately from (6) that

$$KG = S_n, \qquad \text{where} \quad K \in \mathbb{R}\backslash\{0\}.$$

Since a trigonometric polynomial of degree n cannot have $(2n - 1)$ simple real zeros on $[-\pi, +\pi)$, it follows from (6) that for $S^-(S_n) = 2n - 1$,

$$G(-\pi) = S_n(-\pi) = 0 \qquad \text{and thus} \qquad KG = S_n, \text{ where } K \in \mathbb{R}\backslash\{0\}.$$

In view of (5) the theorem is proved. If $a_{n-m} \neq 0$, it follows from (3), by putting $z = 0$, that $d_\nu \in \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, $\nu \in \{1,..., m\}$.

Notation. Let $\tilde{P}_n$, $n \in \mathbb{N}_0$, denote the real trigonometric polynomials of degree equal or less than n. If $R_n \in \tilde{P}_n$ is such that

$$\int_{[-\pi,+\pi]} \frac{|f(\varphi) - R_n(\varphi)|}{s_l(\varphi)}\, d\lambda(\varphi) = \inf_{S_n \in \tilde{P}_n} \int_{[-\pi,+\pi]} \frac{|f(\varphi) - S_n(\varphi)|}{s_l(\varphi)}\, d\lambda(\varphi),$$

we call R_n a best approximation to $f \in L^1[-\pi, +\pi]$ from $\tilde{P}_n$ with respect to the weight function $1/s_l$.

Theorem 2 enables us to give a general representation of the error function when approximating a fixed trigonometric polynomial by trigonometric polynomials of lower degree.

COROLLARY 1. *Suppose $n, m, l \in \mathbb{N}_0$, $n \geqslant m + l + 1$, $3m \leqslant 2n - 2l - 1$. Let S_n be a trigonometric polynomial of degree n and R_{n-m-1} be the best* *approximation to S_n from $\tilde{P}_{n-m-1}$ with respect to the weight function $1/s_l$. If $S_n - R_{n-m-1}$ has $2(n-k)$, $k \in \{0,..., m\}$, simple zeros on $[-\pi, +\pi)$, then there exists a polynomial $\prod_{\nu=1}^{m-k} (z - d_\nu)$, $d_\nu \in \{z \in \mathbb{C} \mid |z| < 1\}$, and a trigonometric polynomial $t_k \in \tilde{P}_k$, which is nonnegative on $[-\pi, +\pi)$, such that*

$$\frac{S_n(\varphi) - R_{n-m-1}(\varphi)}{s_l(\varphi)} = \pm \frac{\mathscr{T}_{n-k,s_l}(\varphi, \prod_{\nu=1}^{m-k} (z - d_\nu))\, t_k(\varphi)}{s_l(\varphi)}$$

(where the leading coefficients of $\mathscr{T}_{n-k,s_l}(\varphi, \prod_{\nu=1}^{m-k} (z - d_\nu))$ have to be chosen suitably).

Proof. Since $S_n - R_{n-m-1}$ is a polynomial of degree n which has exactly $2(n-k)$ simple zeros on $[-\pi, +\pi)$, $S_n - R_{n-m-1}$ can be represented as

$$S_n - R_{n-m-1} = V_{n-k} Z_k,$$

where V_{n-k} is a trigonometric polynomial of degree $n - k$ which has exactly $2(n-k)$ simple zeros on $[-\pi, +\pi)$, and Z_k is a trigonometric polynomial of degree k which is nonpositive or nonnegative on $[-\pi, +\pi)$. Therefore

$$\int_{-\pi}^{+\pi} \frac{\begin{Bmatrix} \sin j\varphi \\ \cos j\varphi \end{Bmatrix}}{s_l(\varphi)} \operatorname{sgn} V_{n-k}(\varphi)\, d\varphi = 0$$

for $j \in \{0,..., (n-k) - (m-k) - 1\}$ (see, e.g., [8, Corollary 1, p. 105]). Applying Theorem 3 to V_{n-k}, the theorem is proved.

Concerning the Solotareff problem for weighted trigonometrical approximation, we need

Notation. Let $s_l(\varphi) = \gamma^2 |\prod_{\nu=1}^{l} (z - (a_\nu + ib_\nu))|^2$, $z = e^{i\varphi}$. For $n \in \mathbb{N}$, $n \geqslant l + 2$, $A, B, \sigma, \tau \in \mathbb{R}$, $R_{n,s_l,A,B}(\cdot, \sigma, \tau)$ denotes that trigonometric rational function which deviates least from zero on $[-\pi, +\pi]$ among all rational functions of the form

$$\Big(A \cos n\varphi + B \sin n\varphi - 2\Big(B \sum_{\nu=1}^{l} b_\nu + A \sum_{\nu=1}^{l} a_\nu + \sigma\Big) \cos(n-1)\,\varphi$$

$$- 2\Big(B \sum_{\nu=1}^{l} a_\nu - A \sum_{\nu=1}^{l} b_\nu + \tau\Big) \sin(n-1)\,\varphi + \cdots\Big)\Big/ s_l(\varphi).$$

The solution of the Solotareff problem follows immediately from Corollary 1. For, if $\sigma^2 + \tau^2 < A^2 + B^2$, we have

$$R_{n,s_l,A,B}(\varphi, \sigma, \tau) = \mathscr{T}_{n,s_l}\Big(\varphi, z - \Big(\frac{(A\sigma + B\tau) + i(B\sigma - A\tau)}{(A^2 + B^2)}\Big)\Big)\Big/ s_l(\varphi),$$

and (Theorem 2(d))

$$\int_{-\pi}^{+\pi} |R_{n,s_l,A,B}(\varphi, \sigma, \tau)| \, d\varphi = \frac{4}{\gamma^2} (A^2 + B^2)^{1/2} \left(1 + \frac{(\sigma^2 + \tau^2)}{(A^2 + B^2)}\right).$$

For $\sigma^2 + \tau^2 \geqslant A^2 + B^2$ one obtains $R_{n,s_l,A,B}(\cdot, \sigma, \tau)$ by multiplication of $T_{n-1}(\cdot, s_l)/s_l$ by a trigonometric polynomial of degree 1, which is positive on $[-\pi, +\pi)$.

Now we consider the representation of the extremal function if, in problem (b), instead of trigonometric polynomials algebraic polynomials are given. As mentioned before, this problem can be regarded as a special case of problem (b).

Notation. $p_l(x) = \prod_{\nu=1}^{l} (1 - x/\alpha_\nu)$ denotes an algebraic polynomial of degree l which is positive on $[-1, +1]$. Furthermore, let $\hat{U}_n(\cdot, p_l) = 2^n U_n(\cdot, p_l)$, where $U_n(\cdot, p_l)$ is defined in [5, p. 36]. Note that $\hat{U}_n(\cdot, 1) = U_n$, where U_n is the Chebyshev polynomial of 2nd type.

DEFINITION 4. Let $\prod_{\nu=1}^{m} (x - d_\nu)$, $m \in \mathbb{N}_0$, be a real polynomial with $d_\nu \in \{z \in \mathbb{C} \mid |z| < 1\}$, where $\prod_{\nu=1}^{0} (x - d_\nu) := 1$. Furthermore, let $\prod_{\nu=1}^{m} (x - d_\nu)^2 = \sum_{\mu=0}^{2m} a_\mu x^\mu$. We define for $n \in \mathbb{N}_0$, $n \geqslant l + m$,

$$\mathscr{U}_{n,p_l}\left(x, \prod_{\nu=1}^{m} (x - d_\nu)\right) := 2^{-n} \sum_{\mu=0}^{2m} a_\mu \hat{U}_{n-2m-\mu}(x, p_l) \qquad (x \in [-1, +1]).$$

We obtain from Definition 4 that $\mathscr{U}_{n,p_l}(\cdot, \prod_{\nu=1}^{m} (x - d_\nu))$ is a polynomial of degree n with leading coefficient 1 and ($\varphi = \arccos x$, $A = 0$, $B = 1$)

$$2^n \mathscr{U}_{n,p_l}\left(x, \prod_{\nu=1}^{m} (x - d_\nu)\right) = \frac{\mathscr{T}_{n+1,p_l(\cos\varphi)}(\varphi, \prod_{\nu=1}^{m} (z - d_\nu))}{\sin \varphi}.$$

Now we formulate Theorem 3, which is the basic result of this paper, for the case of polynomial approximation. Analogously Theorem 2 and Corollary 1 can be transformed.

THEOREM 4. *Let $n \geqslant l + m$, $3m \leqslant 2n + 1 - 2l$. If q_n is a polynomial of degree n with leading coefficient* 1, $S^-(q_n) = n$ *and*

$$\int_{-1}^{+1} \frac{x^k}{p_l(x)} \operatorname{sgn} q_n(x) \, dx = 0, \qquad k \in \{0, \ldots, n - m - 1\},$$

then there exists a real polynomial $\prod_{\nu=1}^{m} (x - d_\nu)$, $d_\nu \in \{z \in \mathbb{C} \mid |z| < 1\}$, such that

$$q_n(x) = \mathscr{U}_{n,p_l}\left(x, \prod_{\nu=1}^{m} (x - d_\nu)\right) \qquad (x \in [-1, +1]).$$

Proof. q_n can be represented in the form $\sum_{i=1}^{n+1} \lambda_i(\sin i \arccos x/\sin \arccos x)$, where $\lambda_i \in \mathbb{R}$, $\lambda_{n+1} = 1/2^n$. If we put $S_{n+1}(\varphi) = \sum_{i=1}^{n+1} \lambda_i \sin i\varphi$, then for $k \in \{1,..., n - m\}$,

$$0 = \int_{-1}^{+1} \frac{U_{k-1}(x)}{p_l(x)} \operatorname{sgn} q_n(x)\, dx = \int_0^{\pi} \frac{\sin k\varphi}{p_l(\cos \varphi)} \operatorname{sgn} S_{n+1}(\varphi)\, d\varphi.$$

Now it follows from Theorem 3, S_{n+1} being a sine polynomial, that

$$2^n S_{n+1}(\varphi) = \mathscr{T}_{n+1, p_l(\cos\varphi)} \left(\varphi, \prod_{\nu=1}^{m} (z - d_\nu)\right), \qquad d_\nu \in \{z \in \mathbb{C} \mid |z| < 1\},$$

where the d_ν are real or complex conjugate. Hence

$$q_n(x) = \frac{S_{n+1}(\varphi)}{\sin \varphi} = \frac{\mathscr{T}_{n+1, p_l(\cos\varphi)}(\varphi, \prod_{\nu=1}^{m} (z - d_\nu))}{2^n \sin \varphi}$$
$$= \mathscr{U}_{n, p_l}\left(x, \prod_{\nu=1}^{m} (x - d_\nu)\right).$$

For $p_l = 1$ Theorem 4 was published by the author in [6]. See also [2, 4]. An application of Theorem 4 gives us the solution of the Solotareff problem for weighted polynomial approximation.

Notation. Put $\alpha_\nu = \frac{1}{2}(c_\nu + 1/c_\nu)$, $c_\nu \in \mathbb{C}$, $|c_\nu| < 1$ for $\nu \in \{1,..., l\}$, and $p_l(x) = \prod_{\nu=1}^{l} (1 - x/\alpha_\nu)$. For $n \in \mathbb{N}$, $n \geqslant l + 1$, $\sigma \in \mathbb{R}$, $r_{n,p_l}(\cdot, \sigma)$ denotes that rational function deviating least from zero on $[-1, +1]$ among all functions of the form

$$\left\{x^n - \left(\sum_{\nu=1}^{l} c_\nu + \sigma\right) x^{n-1} + \sum_{\mu=0}^{n-2} b_\mu x^\mu\right\} \Big/ p_l,$$

with $(b_0, ..., b_{n-2}) \in \mathbb{R}^{n-1}$.

We obtain, from Theorem 4, that

$$p_l \cdot r_{n,p_l}(\cdot, \sigma) = \mathscr{U}_{n,p_l}(\cdot, (x - \sigma)) \qquad \text{for} \quad |\sigma| < 1,$$
$$= (x - \sigma)\, U_{n-1}(\cdot, p_l) \qquad \text{for} \quad |\sigma| \geqslant 1,$$

and from Theorem 2(d), with $\gamma^2 = 1/\prod_1^l (1 + c_\nu^2)$, that

$$\int_{-1}^{+1} |r_{n,p_l}(x, \sigma)|\, dx = 2^{-n+1}(1 + \sigma^2) \prod_{\nu=1}^{l} (1 + c_\nu^2) \qquad \text{for} \quad |\sigma| < 1,$$

$$= 2^{-n+2} |\sigma| \prod_{\nu=1}^{l} (1 + c_\nu^2) \qquad \text{for} \quad |\sigma| \geqslant 1.$$

4. Further Applications

With the aid of the polynomials introduced in Definition 3 we are able to determine the location of the zeros of the error function for the polynomial approximation. For results of this type see also [7].

Notation. For $n \in \mathbb{N}_0$, let P_n denote the real polynomials of degree n or less. Furthermore, let $Z(f) = \{x \in [a, b] \mid f(x) = 0\}$ for $f \in L^1[a, b]$.

Independent of the polynomials introduced above, the following theorem can be shown.

Theorem 5. *Suppose that $f \in C[a, b]$ and that p_{n-1} is the best approximation to f from P_{n-1} on $[-1, +1]$. If $S^-(f - p_{n-1}) \geq n + 1$ and $f - p_{n-1}$ has a finite number of distinct zeros in $[-1, +1]$, then $f - p_{n-1}$ changes sign at least once in each interval $(-\cos(i-1)\pi/(n+1), -\cos i\pi/(n+1))$, $i = 1,..., n+1$.*

Proof. According to Rice [8],

$$\int_{-1}^{+1} x^k [\operatorname{sgn} U_n(x) - \operatorname{sgn}(f - p_{n-1})(x)]\, dx = 0, \qquad k \in \{0,..., n-1\}.$$

Assume there exists a $j \in \{1,..., n+1\}$ such that $f - p_{n-1}$ does not change sign in the interval $(-\cos(j-1)\pi/(n+1), -\cos j\pi/(n+1))$. Then $\operatorname{sgn} U_n(x) \overset{-}{(+)} \operatorname{sgn}(f - p_{n-1})(x) = 0$ for $x \in (-\cos(j-1)\pi/(n+1), -\cos j\pi/(n+1))$, from which we can conclude that $\operatorname{sgn} U_n \overset{-}{(+)} \operatorname{sgn}(f - p_{n-1})$ has at most $(n-1)$ changes of sign on $(-1, +1)$. From Lemma 2(a) it follows now that $\operatorname{sgn} U_n = \overset{+}{(-)} \operatorname{sgn}(f - p_{n-1})$. This is in contradiction to $S^-(f - p_{n-1}) \geq n + 1$.

Lemma 3 (Meinardus [5, p. 34]). *The zeros $(-1 <)\, x_1(d) < x_2(d) < \cdots < x_n(d) (< 1)$ of the polynomial $2^n \mathscr{U}_{n,1}(\cdot, x - d) = U_n - 2dU_{n-1} + d^2 U_{n-2}$ are increasing with respect to $d \in (-1, +1)$. Furthermore, $x_i(0) = -\cos i\pi/(n+1)$, $i = 1, 2,..., n$.*

Proof. Since $U_n(x_i(d)) - 2dU_{n-1}(x_i(d)) + d^2 U_{n-2}(x_i(d)) = 0$,

$$\left(x_i(d) - \frac{2d}{1 + d^2}\right) T_n'(x_i(d)) + n \left(\frac{1 - d^2}{1 + d^2}\right) T_n(x_i(d)) = 0,$$

where T_n denotes the Chebyshev polynomial of the first kind. If we put $d = (1 - (1 - \tau^2)^{1/2})/\tau$ for $\tau \in (-1, +1)$, $\tau \neq 0$, it follows from [5, p. 34] that

$$\frac{dx_i}{d\tau} = \frac{1 - x_i^2}{n(1 - \tau^2)^{1/2}(1 - \tau x_i) + (1 - \tau^2)} > 0.$$

THEOREM 6. *Let* $f \in C^{n+1}[-1, +1]$, $f^{(n+1)}(x) > 0$ *for* $x \in [-1, +1]$, *let* $p_n = a_n x^n + \cdots$ *be its best approximation from* P_n *and* $p_{n-1} \neq p_n$ *its best approximation from* P_{n-1} *with* $S^-(f - p_{n-1}) = n + 1$. *If* $a_n > 0$ (<0), *then* $f - p_{n-1}$ *changes sign exactly once in each interval* $(-\cos i\pi/(n+2), -\cos i\pi/(n+1))((-\cos(i-1)\pi/(n+1), -\cos i\pi/(n+2)))$, $i = 1,..., n+1$.

Proof. Since p_n and p_{n-1} are best approximations to f, it follows from [9, p. 120] and [8, p. 105] that

$$0 < \int_{-1}^{+1} (p_{n-1}(x) - p_n(x)) \operatorname{sgn}(f - p_{n-1})(x)\, dx$$

$$= -a_n \int_{-1}^{+1} x^n \operatorname{sgn}(f - p_{n-1})(x)\, dx.$$

Using the fact that there exists a $d^* \in (-1, +1)\backslash\{0\}$ such that $\operatorname{sgn}(f - p_{n-1})(x) = \pm\operatorname{sgn} \mathscr{U}_{n+1,1}(x, (x - d^*))$ and, in fact, $\operatorname{sgn}(f - p_{n-1})(x) = \operatorname{sgn} \mathscr{U}_{n+1,1}(x, (x - d^*))$, since $f^{(n+1)} > 0$, we get

$$0 < -\frac{a_n}{2^n} \int_{-1}^{+1} U_n(x) \operatorname{sgn} \mathscr{U}_{n+1,1}(x, (x - d^*))\, dx$$

$$= -\frac{a_n}{2^n \pi i} \int_{|z|=1} z^{-1} \frac{(z - d^*)}{(1 - d^* z)}\, dz = \frac{a_n d^*}{2^{n-1}}.$$

Hence $\operatorname{sgn} a_n = \operatorname{sgn} d^*$. According to Lemma 3 and Theorem 5 the theorem is proved.

Note that since $f^{(n+1)}(x) > 0$ for $x \in [-1, +1]$, p_n is that polynomial which interpolates f at the zeros of U_{n+1}. Therefore the leading coefficient a_n of p_n can be determined quite simply. Finally it should be noted that with the help of the polynomials $\mathscr{U}_{n,p_l}(\cdot, \prod_{\nu=1}^{m} (x - d_\nu))$, sufficient conditions can be stated for the uniqueness of the best weighted (weight $1/p_l$) polynomial approximation to a piecewise continuous function with jumps. For $p_l = 1$ see [6].

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